Algebra Qualifying Exam I (January 2023)

You have 120 minutes to complete this exam.

1. (10 points) Let G be a group of order 81. Is G necessarily abelian? Please justify your conclusion.

2. (10 points) Let S_n be the symmetric group with $n \geq 5$. Find all the normal subgroups of S_n. Please justify your conclusion.

3. (10 points) Determine the greatest common divisor of the polynomials $f(x) = x^{100} - 1$ and $g(x) = x^{17} + 2x^6 + x + 100$ in the polynomial ring $\mathbb{Q}[x]$.

4. (10 points) Consider the ring

$$\mathbb{Z}[\sqrt{-2}] = \left\{ a + b\sqrt{-2} \;\middle|\; a, b \in \mathbb{Z} \right\}.$$

   Define $N(a + b\sqrt{-2}) = a^2 + 2b^2$. Prove that $\mathbb{Z}[\sqrt{-2}]$ with the function N forms a Euclidean domain.

5. (15 points) Let $\mathbf{F}_7$ be the finite field of 7 elements.
   (a) Prove that the polynomial $f(x) = x^7 - x + 1$ is irreducible in $\mathbf{F}_7[x]$.
   (b) Prove that the polynomial $g(x) = x^7 + 7x^5 + 6x + 8$ is irreducible in $\mathbb{Z}[x]$.

6. (10 points) Let $\alpha = \cos\frac{4\pi}{11}$. Let $E = \mathbb{Q}(\alpha)$. Compute the Galois group $\mathrm{Gal}(E/\mathbb{Q})$.